Gallai-Ramsey Number of Graphs

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May 2019

Outline

1. Introduction
   - Classical Ramsey number
   - Gallai Ramsey number under K_3-coloring
   - Gallai Ramsey number under S_3^+-coloring
2. Main results
   - Books
   - Wheels
   - Fans
   - Stars with extra independent edges
   - Odd cycles
   - Two Classes of Unicyclic Graphs
   - Results for S_3^+-coloring

Co-authors

- Joint work with

  Colton Magnant: Clayton State University, USA;

  Ingo Schiermeyer: Technische Universität Bergakademie Freiberg, Germany;

  Zhao Wang: China Jiliang University, China;

  Jinyu Zou: Qinghai Normal University, China.

Outline

Rainbow coloring

- A coloring of a graph is called rainbow if no two edges have the same color.
- Colorings of complete graphs that contain no rainbow triangle have very interesting and somewhat surprising structure.
- T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hungar* 18 (1967), 25–66 first examined this structure under the guise of transitive orientations.

Rainbow coloring

- The result was reproven in A. Gyárfás and G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory* 46(3) (2004), 211–216 in the terminology of graphs and can also be traced to K. Cameron and J. Edmonds, Lambda composition, *J. Graph Theory* 26(1) (1997), 9–16.

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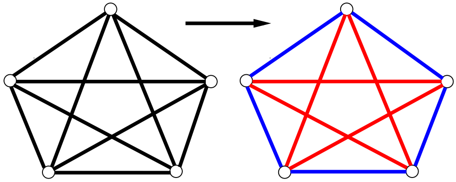
Classical Ramsey number

- Given two graphs G and H, the graph Ramsey number $R(G,H)$ is the minimum integer n such that every (red/blue)-coloring of the edges of K_n contains either a red copy of G or a blue copy of H.



K_5 Colored K_5

Classical Ramsey number

- $R(K_3, K_3) = 6$;

  $R(K_4, K_4) = 18$;

  $43 \leq R(K_5, K_5) \leq 49$;

  $\cdots$

- See the dynamic survey S. P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.*, Dynamic Survey 1, 30, 1994.

Outline

Gallai-Ramsey number

Definition 1

Given two graphs G and H and an integer k, the Gallai-Ramsey number $gr_k(G : H)$ is defined to be the minimum integer n such that any k coloring of the complete graph K_n contains either a rainbow colored G or a monochromatic H.

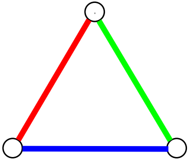
- We generally assume $G = K_3$, and therefore $k \geq 3$.
- Note that these numbers are bounded by multicolor Ramsey numbers so existence is obvious.

Rainbow triangle free coloring

- For the following statement, a trivial partition is a partition into only one part.

Theorem 1.1

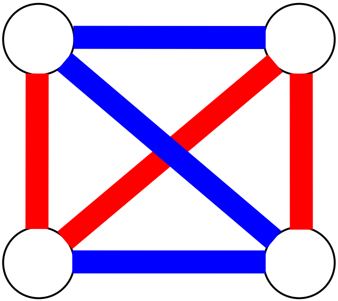
The vertices of every rainbow triangle free coloring of a complete graph can be partitioned such that all edges between a pair of parts have a single color and all edges between the parts come from only two colors.



- Here by "rainbow" we mean all different colors.

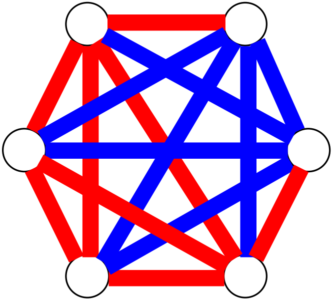
Gallai Partition under K_3-coloring

- What a partition might look like.



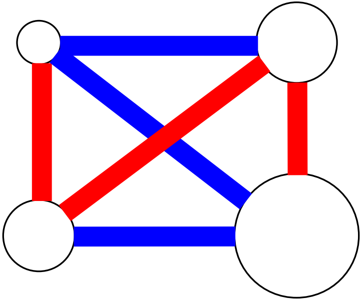
Gallai Partition under K_3-coloring

- But we don't know how many parts there are...



Gallai Partition under K_3-coloring

- But we don't know how many parts there are...or big they are...



Gallai Partition under K_3-coloring

- But we don't know how many parts there are... or big they are... or what happens inside the parts...

Gallai Partition under K_3-coloring

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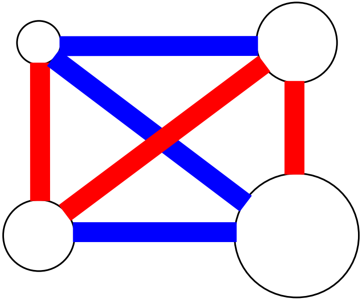
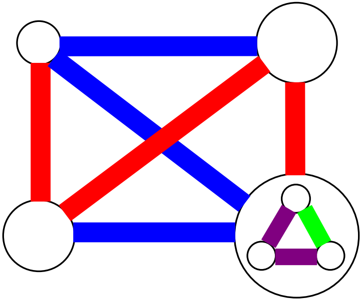


Gallai Partition under K_3-coloring

- But we don't know how many parts there are... or big they are... or what happens inside the parts... but we do know the inside is rainbow triangle free.



Gallai-coloring under K_3-coloring

- For ease of notation, we refer to a colored complete graph with no rainbow triangle as a Gallai-coloring and the partition provided by Theorem 1.1 as a Gallai-partition.
- The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the reduced graph of that partition.
- By Theorem 1.1, the reduced graph is a 2-colored complete graph.

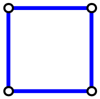
Survey papers

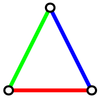
- The theory of Gallai-Ramsey numbers has grown by leaps and bounds in recent years, especially for the case where $G = K_3$.
- We refer the interested reader to the survey S. Fujita, C. Magnant, K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* 26(1) (2010), 1–30.
- An updated version at S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of Ramsey theory–a dynamic survey, *Theo. Appl. Graphs* 0(1), 2014 for more general information.

Outline of a Simple Proof

Theorem 1.2

$$gr_k(K_3 : C_4) = k + 4.$$

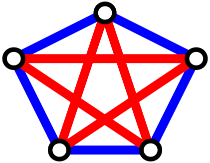




Outline of a Simple Proof

- For the lower bound, consider the lex. coloring, starting at a 2-colored K_5.

- No rainbow triangle and no monochromatic C_4.



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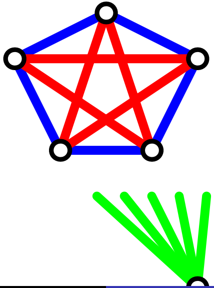

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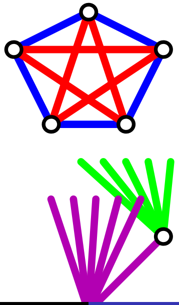
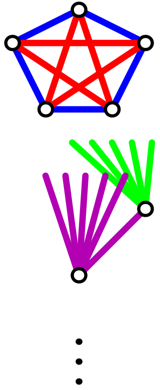


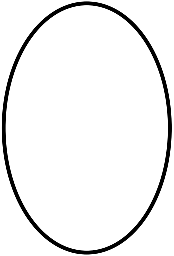
Outline of a Simple Proof

- $n = k + 3$, no rainbow triangle, no monochromatic C_4.



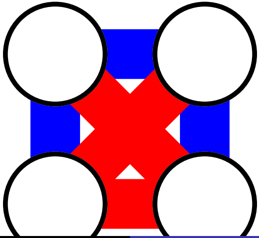
Outline of a Simple Proof

- Now consider an arbitrary coloring of K_{k+4} using k colors with no rainbow triangle.



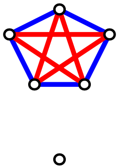
Outline of a Simple Proof

- There is a Gallai partition, but we don't know how big the parts are.
- Recall: all one color between each pair.



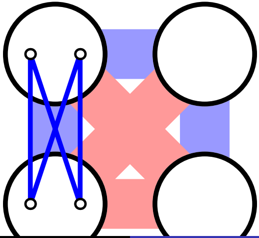
Outline of a Simple Proof

- If all parts are single vertices, then we simply have a 2-coloring with $n \geq 6$.
- Applying $R(C_4, C_4) = 6$, we have



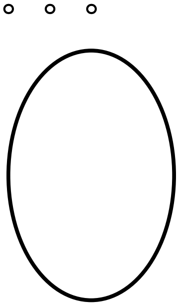
Outline of a Simple Proof

- There must be parts with at least 2 vertices.
- But if two parts have at least 2 vertices each, then we've easily gotten a monochromatic C_4.



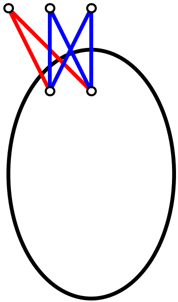
Outline of a Simple Proof

- Then there must be at most one big part (≥ 2), and all others are one vertex.



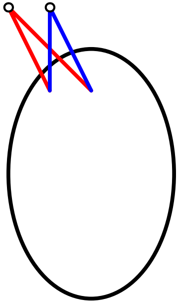
Outline of a Simple Proof

- If there are three vertices outside, then pigeonhole gives us a C_4.



Outline of a Simple Proof

- Finally there are at most two vertices outside, induct on the number of colors (with maximum degree at least 2) in the big part.



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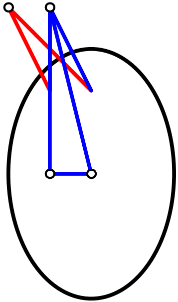

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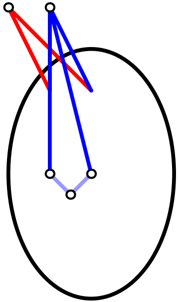

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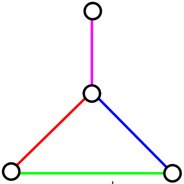
Gallai Partition under S_3^+-coloring

- Let S_3^+ be the graph on 4 vertices consisting of a triangle and a pendant edge.
- S. Fujita, C. Magnant, Extensions of Gallai-Ramsey results, J. Graph Theory 70(4) (2012), 404–426 proved a decomposition theorem for rainbow S_3^+-free colorings of a complete graph.



S_3^+

Gallai Partition under S_3^+-coloring

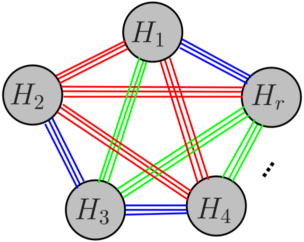
Theorem 1.3 (Fujita, Magnant)

In any rainbow S_3^+-free coloring G of a complete graph, one of the following holds:

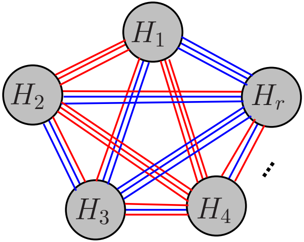
(1) *$V(G)$ can be partitioned such that there are 2 colors on the edges among the parts, and at most 2 colors on the edges between each pair of parts; or*

(2) *There are three (different colored) monochromatic spanning trees, and moreover, there exists a partition of $V(G)$ with exactly 3 colors on edges between parts and between each pair of parts, the edges have only one color.*

Gallai Partition under S_3^+-coloring



Type 1



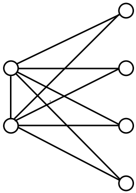
Type 2

Outline

Outline

The book graph

- The book graph with m pages is denoted by B_m, where $B_m = K_2 + K_m$.
- Note that $B_1 = K_3$ and $B_2 = K_4 \backslash \{e\}$ where e is an edge of the K_4.
- In this work, we prove bounds on the Gallai-Ramsey number of all books, with sharp results for several small books.

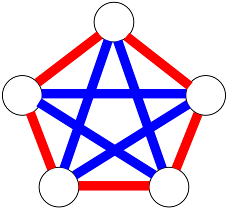


4

Small graphs

Theorem 2.1 (Gyarfas, Simonyi (2004))

In any G-coloring of a complete graph, there is a vertex with at least $\frac{2n}{5}$ incident edges in a single color.



Theorem 2.2 (Chvátal and Harary (1976), Rousseau and Sheehan (1978))

$R(B_2, B_2) = 10$, $R(B_3, B_3) = 14$, $R(B_4, B_4) = 18$, $R(B_5, B_5) = 21$.

General Lower Bound

- In this section, we prove a lower bound on the Gallai-Ramsey number for books by a straightforward inductive construction.

General Lower Bound

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Theorem 2.3 (Zou, Mao, Wang, Magnant, Ye)

If B_m is the book with m pages, $B_m = K_2 + \overline{K_m}$, then for $k \geq 2$,

$$gr_k(K_3 : B_m) \geq \begin{cases} (R(B_m, B_m) - 1) \cdot 5^{(k-2)/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot (R(B_m, B_m) - 1) \cdot 5^{(k-3)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

General Upper Bound

- Let $R_m = R(B_m : B_m)$ and define

$$R'_m = \sum_{i=1}^{m-1} [R_{\lceil m/i \rceil} - 1].$$

This quantity provides a bound on a type of restricted Ramsey number as seen in the following lemma.

Lemma 2

For $m \geq 2$, the largest number of vertices in a G-coloring of a complete graph with no monochromatic B_m in which all parts of the G-partition have order at most $m-1$ is at most R'_m.

General Upper Bound

- Note that for m large, since $R_m \sim (4 + o(1))m$ (see [Rousseau and Sheehan (1978)]), we get

$$R'_m \sim (4 + o(1))m \ln[(4 + o(1))m].$$

For small values of m, we compute $R'_2 = 9$, $R'_3 = 22$, and $R'_4 = 35$.

- Call a color *m-admissible* if it induces a subgraph with maximum degree at least m, and *m-inadmissible* otherwise.

General Upper Bound

Lemma 3 (Zou, Mao, Wang, Magnant, Ye)

Given integers $m \geq 2$ and $k \geq 2$, let n be the largest number of vertices in a k-coloring of a complete graph in which there is

- *no rainbow triangle,*
- *no monochromatic B_m,*
- *a G-partition with all parts having order at most $m-1$, and*
- *only one m-admissible color.*

Then

$$n \leq \begin{cases} 3m-1 & \text{if } k=2, \\ 5m-5 & \text{otherwise.} \end{cases}$$

General Upper Bound

- Let $\ell = \ell(m)$ be the number of colors that are m-inadmissible and define the quantity $gr_{k,\ell}(K_3 : H)$ to be the minimum integer n such that every k coloring of K_n with at least ℓ different m-inadmissible colors contains either a rainbow triangle or a monochromatic copy of H.
- We may now state our main result, which provides a general upper bound on the Gallai-Ramsey numbers for any book with any number of colors.

General Upper Bound

Theorem 2.4 (Zou, Mao, Wang, Magnant, Ye)

Given positive integers $k \geq 1$, $m \geq 3$, *and* $0 \leq \ell \leq k$, *let*

$$gr_{k,\ell,m} = \begin{cases} m + 2 - \ell & \text{if } k = 1, \\ R'_m \cdot 5^{\frac{k-2}{2}} + 1 - (m-1)\ell & \text{if } k \text{ is even}, \\ 2 \cdot R'_m \cdot 5^{\frac{k-3}{2}} + 1 - (m-1)\ell & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Then

$$gr_{k,\ell}(K_3 : B_m) \leq gr_{k,\ell,m}.$$

Some Small Cases

- In this section, we provide the sharp Gallai-Ramsey number for several small books. The proof of this result follows the proof of Theorem 2.23 except each step is improved in order to produce the sharp result.

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Lemma 4

Let k, ℓ, m be integers with $k \geq 3$, $0 \leq \ell \leq k-2$, and $2 \leq m \leq 5$. If G is a G-coloring of K_p with

$$p \geq gr_{k,\ell,m}$$

using k colors in which all parts of a G-partition have order at most $m-1$ and ℓ colors are m-inadmissible, then G contains a monochromatic copy of B_m.

Some Small Cases

- For the values of m in question, we have

$$p \geq \begin{cases} 18 & \text{if } m = 2, \\ 25 & \text{if } m = 3, \\ 32 & \text{if } m = 4, \text{ and} \\ 37 & \text{if } m = 5. \end{cases}$$

- Let t be the number of parts in the partition. When $m = 2$, all parts of the assumed G-partition have order 1, meaning that G is simply a 2-coloring. Since $|G| = p > 10 = R_2$, the claim is immediate. We consider cases for the remaining values of m.

Outline

Gallai-Ramsey number for wheels

- Let W_n be a wheel of order n, that is, $W_n = K_1 \vee C_{n-1}$ where C_{n-1} is the cycle on $n-1$ vertices.

Theorem 2.5 (Mao, Wang, Magnant, Schiermeyer)

$$(1)\ R(W_5, W_5) = 15;$$
$$(2)\ R(W_6, W_6) = 17.$$

Gallai-Ramsey number for wheels

- As far as we are aware, for $n \geq 7$, the classical diagonal Ramsey number for the wheel is yet unknown.
- We give upper and lower bounds for classical Ramsey number of the general wheel W_n.

Theorem 2.6 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 1$ and $n \geq 7$,

$$\begin{cases} 3n-3 \leq R(W_n, W_n) \leq 8n-10, & \text{if } n \text{ is even;} \\ 2n-2 \leq R(W_n, W_n) \leq 6n-8 & \text{if } n \text{ is odd.} \end{cases}$$

Gallai-Ramsey number for wheels

- We obtain the exact value of the Gallai Ramsey number for W_5.

Theorem 2.7 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 1$,

$$gr_k(K_3 : W_5) = \begin{cases} 5 & \text{if } k = 1, \\ 14 \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even}, \\ 28 \cdot 5^{\frac{k-3}{2}} + 1 & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Gallai-Ramsey number for wheels

- We provide general lower bounds on the Gallai-Ramsey numbers for all wheels.

Theorem 5 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 2$ and $n \geq 6$, we have

$$gr_k(K_3 : W_n) \geq \begin{cases} (3n-4)5^{\frac{k-2}{2}}+1 & \text{if } n \text{ is even and } k \text{ is even;} \\ (6n-8)5^{\frac{k-3}{2}}+1 & \text{if } n \text{ is even and } k \text{ is odd;} \\ (2n-3)5^{\frac{k-2}{2}}+1 & \text{if } n \text{ is odd and } k \text{ is even;} \\ (4n-6)5^{\frac{k-3}{2}}+1 & \text{if } n \text{ is odd and } k \text{ is odd.} \end{cases}$$

Gallai-Ramsey number for wheels

- We provide general upper bounds on the Gallai-Ramsey numbers for all wheels.

Theorem 6 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 3$ and $n \geq 6$, we have

$$gr_k(K_3 : W_n) \leq (n-4)^2 \cdot 30^k + k(n-1).$$

Outline

Gallai-Ramsey number for fans

- The fan graph of order n is denoted by F_n, where $B_n = K_1 + n\overline{K_2}$.
- Note that $F_1 = K_3$ and F_2 is a graph obtained from two triangles by sharing one vertex.

Theorem 2.8

(1) $R(F_2, F_2) = 9$;
(2) $R(F_3, F_3) = 13$;
(3) $4n + 1 \leq R(F_n, F_n) \leq 6n$ *for large sufficiently* n, $6n < n^2 + 1$.

Gallai-Ramsey number for fans

- First our sharp result for F_2.

Theorem 2.9 (Mao, Wang, Magnant, Schiermeyer)

$$gr_k(K_3; F_2) = \begin{cases} 9, & \text{if } k = 2; \\ \frac{83}{2} \cdot 5^{\frac{k-4}{2}} + \frac{1}{2}, & \text{if } k \textit{ is even}, \ k \geq 4; \\ 4 \cdot 5^{\frac{k-1}{2}} + 1, & \text{if } k \textit{ is odd}. \end{cases}$$

Gallai-Ramsey number for fans

- Next our general bounds (and sharp result for any even number of colors) for F_3.

Theorem 2.10 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 2$,

$$\begin{cases} gr_k(K_3;F_3) = 14 \cdot 5^{\frac{k-2}{2}} - 1, & \text{if } k \text{ is even;} \\ gr_k(K_3;F_3) = 33 \cdot 5^{\frac{k-3}{2}}, & \text{if } k = 3,5; \\ 33 \cdot 5^{\frac{k-3}{2}} \leq gr_k(K_3;F_3) \leq 33 \cdot 5^{\frac{k-3}{2}} + \frac{3}{4} \cdot 5^{\frac{k-5}{2}} - \frac{3}{4}, & \text{if } k \text{ is odd, } k \geq 7. \end{cases}$$

Gallai-Ramsey number for fans

- In particular, we conjecture the following, which claims that the lower bound in above theorem is the sharp result.

Conjecture 2.1 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 2$,

$$gr_k(K_3; F_3) = \begin{cases} 14 \cdot 5^{\frac{k-2}{2}} - 1, & \textit{if } k \textit{ is even;} \\ 33 \cdot 5^{\frac{k-3}{2}}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Gallai-Ramsey number for fans

- Finally our general bound for all fans.

Theorem 2.11 (Mao, Wang, Magnant, Schiermeyer)

For $k \geq 2$,

$$\begin{cases} 4n \cdot 5^{\frac{k-2}{2}} + 1 \leq gr_k(K_3; F_n) \leq 10n \cdot 5^{\frac{k-2}{2}} - \frac{5}{2}n + 1, & \text{if } k \text{ is even;} \\ 2n \cdot 5^{\frac{k-1}{2}} + 1 \leq gr_k(K_3; F_n) \leq \frac{9}{2}n \cdot 5^{\frac{k-1}{2}} - \frac{5}{2}n + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Outline

Stars with extra independent edges

- Let S_t^r be a star of order t by adding extra r independent edges for $0 \leq r \leq \frac{t-1}{2}$.
- For $r = 0$ we obtain $S_t^r = K_{1,t-1}$, which are called stars.
- For $r = \frac{t-1}{2}$ if t is odd we obtain $S_t^r = F_{\frac{t-1}{2}}$, which are called *fans*.

Results for Ramsey number

- We deal with those graphs S_t^r, where $0 < r < \frac{t-1}{2}$, i.e. where S_t^r is neither a star nor a fan.

Theorem 2.12 (Mao, Wang, Magnant, Schiermeyer)

(1) *For* $t \geq 7$, $R(S_t^2, S_t^2) = 2t - 1$.
(2) *For* $t \geq 15$, $R(S_t^3, S_t^3) = 2t - 1$.
(3) *For* $t \geq 6r - 5$, $R(S_t^r, S_t^r) = 2t - 1$.

Results for Ramsey number

- For graph S_t^2, we have the following.

Theorem 2.13 (Mao, Wang, Magnant, Schiermeyer)

(1) *For $k \geq 1$,*

$$gr_k(K_3; S_6^2) = \begin{cases} 2 \times 5^{\frac{k}{2}} + \frac{1}{4} \times 5^{\frac{k-2}{2}} + \frac{3}{4}, & \text{if } k \text{ is even;} \\ \lceil \frac{51}{10} \times 5^{\frac{k-1}{2}} + \frac{1}{2} \rceil, & \text{if } k \text{ is odd.} \end{cases}$$

(2) *For $k \geq 3$,*

$$gr_k(K_3; S_8^2) = \begin{cases} 14 \times 5^{\frac{k-2}{2}} + \frac{1}{2} \times 5^{\frac{k-4}{2}} + \frac{1}{2}, & \text{if } k \text{ is even;} \\ 7 \times 5^{\frac{k-1}{2}} + \frac{1}{4} \times 5^{\frac{k-3}{2}} + \frac{3}{4}, & \text{if } k \text{ is odd.} \end{cases}$$

Results for Gallai-Ramsey number

- For graph S_t^2, we have the following.

Theorem 2.14 (Mao, Wang, Magnant, Schiermeyer)

(3) *For $k \geq 1$ and $t \geq 6$,*

$$\begin{cases} 2(t-1) \times 5^{\frac{k-2}{2}} + 1 \leq gr_k(K_3; S_t^2) \leq 2t \times 5^{\frac{k-2}{2}}, & \text{if } k \text{ is even;} \\ (t-1) \times 5^{\frac{k-1}{2}} + 1 \leq gr_k(K_3; S_t^2) \leq t \times 5^{\frac{k-1}{2}}, & \text{if } k \text{ is odd.} \end{cases}$$

Results for Gallai-Ramsey number

- For graph S_t^r, we have the following.

Theorem 2.15 (Mao, Wang, Magnant, Schiermeyer)

For $t \geq 6r-5$, if k is even, then

$$2(t-1) \times 5^{\frac{k-2}{2}} + 1 \leq gr_k(K_3; S_t^r) \leq [2t + 8(r-1)] \times 5^{\frac{k-2}{2}} - 4(r-1);$$

If k is odd, then

$$(t-1) \times 5^{\frac{k-1}{2}} + 1 \leq gr_k(K_3; S_t^r) \leq [t + 4(r-1)] \times 5^{\frac{k-1}{2}} - 4(r-1).$$

Outline

Upper and lower bound

- S. Fujita, C. Magnant, Gallai-Ramsey numbers for cycles, *Discrete Math.* 311(13)(2011), 1247–1254 and M. Hall, C. Magnant, K. Ozeki, and M. Tsugaki. Improved upper bounds for Gallai-Ramsey numbers of paths and cycles, *J. Graph Theory* 75(1)(2014), 59–74 derived the following result.

Theorem 2.16

$$\ell 2^k + 1 \leq gr_k(K_3 : C_{2\ell+1}) \leq \ell(2^{k+3} - 3)\log \ell.$$

- Sadly, we were not clever enough to get closer. We believe the lower bounds to be the truth...

For C_3

- For the triangle $C_3 = K_3$, M. Axenovich, P. Iverson, Edge-colorings avoiding rainbow and monochromatic subgraphs, *Discrete Math.* 308(20)(2008), 4710–4723 and F. R. K. Chung, R. L. Graham. Edge-colored complete graphs with precisely colored subgraphs, *Combinatorica* 3(3-4)(1983), 315–324 and A. Gyárfás, G. Sárközy, A. Sebő, S. Selkow, Ramsey-type results for gallai colorings, *J. Graph Theory* 64(3)(2010), 233–243 obtained the following result.

Theorem 2.17

For $k \geq 2$,

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

For C_5

- For C_5, S. Fujita, C. Magnant, Gallai-Ramsey numbers for cycles, *Discrete Math.* 311(13)(2011), 1247–1254 obtained the following result.

Theorem 2.18

For any positive integer $k \geq 2$, we have

$$gr_k(K_3 : C_5) = 2^{k+1} + 1.$$

Our Main Result

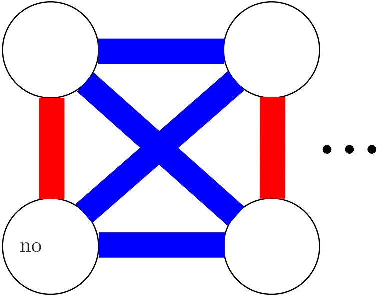
Theorem 2.19 (Wang, Mao, Magnant, Schiermeyer, Zou)

For integers $\ell \geq 3$ and $k \geq 1$, we have

$$gr_k(K_3 : C_{2\ell+1}) = \ell \cdot 2^k + 1.$$

- The lower bound is sharp for odd cycles!

General Lower Bound for Odd Cycles

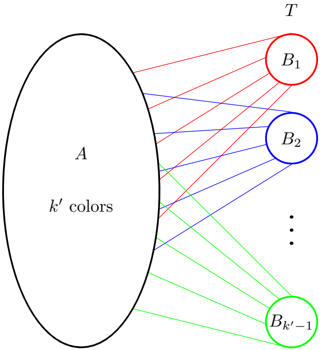


Proof Outline - Setup

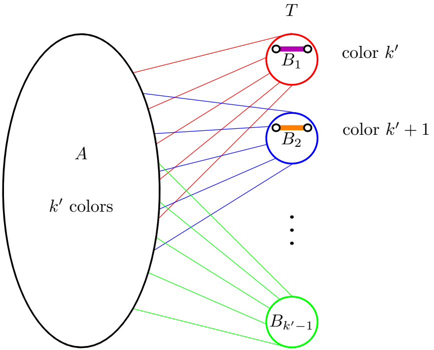
Upper bound proof setup:

- Induction on the number of colors k.
- First set aside vertices with (almost) all one color on their incident edges, call the set T.
- Claim: T is not too big.
- If T is big, then there is a large set of vertices $B_i \subseteq T$ (say $|B_i| \geq \ell$) with all color i to what remains (A).
- That color must be missing from A, so apply induction.

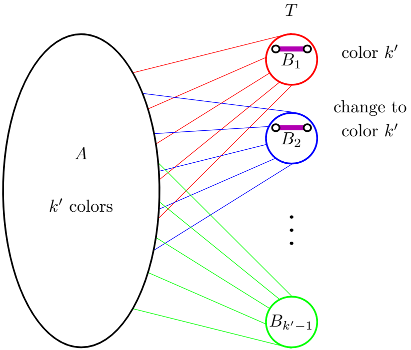
Main Lemma Preview



Main Lemma Preview



Main Lemma Preview



Proof Outline - Main Lemma

Lemma 7

Let $k \geq 3$, $2 \leq k' \leq k$ and let G be a Gallai coloring of the complete graph K_n containing no monochromatic copy of $C_{2\ell+1}$.
If $G = A \cup B_1 \cup B_2 \cup \cdots \cup B_{k'-1}$ where A uses at most k' colors, $|B_i| \leq 2\ell$ for all i, and all edges between A and B_i have color i, then $n \leq gr_{k'}(K_3 : H) - 1$.

- Note that this lemma uses the assumed structure to provide a bound on $|G|$ even if G itself uses more than k' colors.

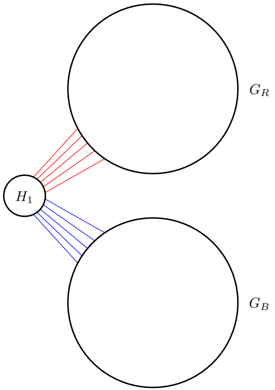
Proof Outline - Reductions

Proof outline:

- Claims: The largest part of partition is small ($|H_1| \leq \ell/2$).
- This means $k = 3$ since no $C_{2\ell+1}$ could fit inside a part, so $n = 8\ell + 1$.
- Finally consider structure of H_1 relative to the rest of G.

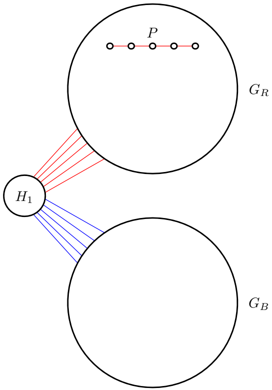
Sketch of the Proof

- Broad structure of G. Suppose G_R is the larger side.



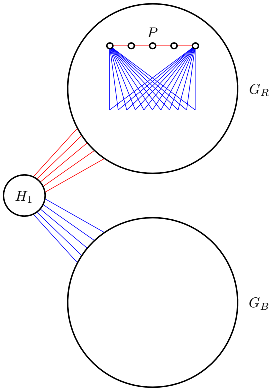
Sketch of the Proof

- Let P be a longest red path within G_R.



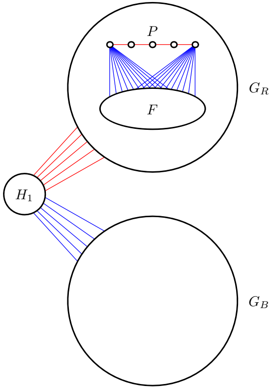
Sketch of the Proof

- The ends cannot have more red edges to the rest of G_R.



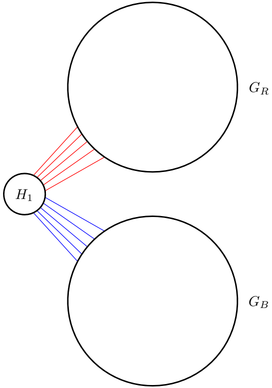
Sketch of the Proof

- Within this remaining set F, there can be no long red path and no long blue path.



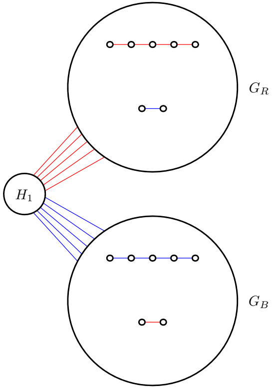
Sketch of the Proof

- The two sides are roughly balanced: $|G_R| \sim |G_B|$.



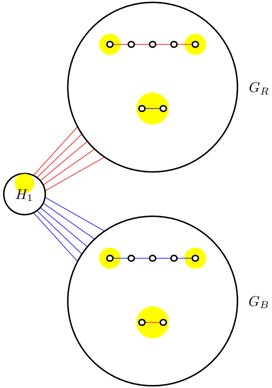
Sketch of the Proof

- Symmetry and similar ideas show we have this structure.



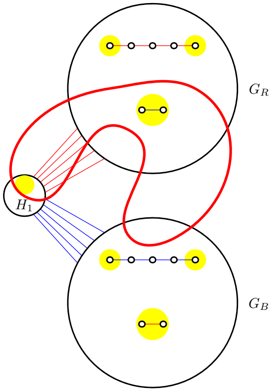
Sketch of the Proof

- Reserve several key vertices for later use. (Absorbing argument)



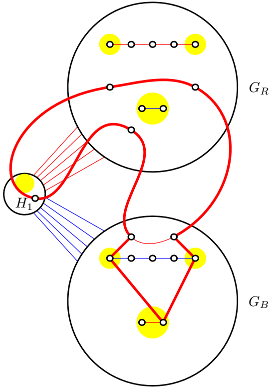
Sketch of the Proof

- Apply the known even cycles result to obtain a mono-chromatic copy of $C_{2\ell-2}$ avoiding the reserved set.



Sketch of the Proof

- Construct a monochromatic copy of $C_{2\ell+1}$ using the reserved set.



Outline

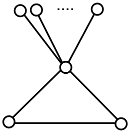
Two Classes of Unicyclic Graphs

In this work, we consider the Gallai-Ramsey numbers for finding either a rainbow triangle or monochromatic graph coming from two classes of unicyclic graphs:

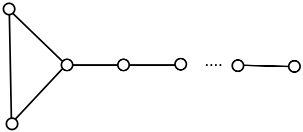
- a star with an extra edge that forms a triangle, and
- a path with an extra edge from an end vertex to an internal vertex formaing a triangle.

Graphs S_t^+ and P_t^+

- Let S_t^+ denote graph consisting of the star S_t with the addition of an edge between two of the pendant vertices, forming a triangle.
- Let P_t^+ denote the graph consisting of the path P_t with the addition of an edge between one end and the vertex at distance 2 along the path from that end, forming a triangle.



S_t^+



P_t^+

Two Classes of Unicyclic Graphs

- These graphs are particularly interesting because although they are not bipartite, they are very close to being a tree (and therefore bipartite).
- The dichotomy between bipartite and non-bipartite graphs is critical in the study of Gallai-Ramsey numbers in light of the following result.

General Upper Bound

- A. Gyárfás, G. Sárközy, A. Sebő, and S. Selkow. Ramsey-type results for gallai colorings, *J. Graph Theory* 64(3)(2010), 233–243 obtained the following result.

Theorem 2.20

Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $gr_k(K_3 : H)$ is exponential in k. If H is bipartite, then $gr_k(K_3 : H)$ is linear in k.

Two Classes of Unicyclic Graphs

- In order to produce sharp results for the Gallai-Ramsey numbers of these graphs, we first proved the 2-color Ramsey numbers for these graphs.

Theorem 2.21 (Wang, Mao, Magnant, Zou)

For $t \geq 3$,

$$R(S_t^+, S_t^+) = 2t - 1.$$

Two Classes of Unicyclic Graphs

- In order to produce sharp results for the Gallai-Ramsey numbers of these graphs, we first proved the 2-color Ramsey numbers for these graphs.

Theorem 2.22 (Wang, Mao, Magnant, Zou)

For $t \geq 4$,

$$R(P_t^+, P_t^+) = 2t - 1.$$

Gallai-Ramsey number for S_t^+

- The precise Gallai-Ramsey number for S_t^+ are obtained.

Theorem 2.23 (Wang, Mao, Magnant, Zou)

For $k \geq 1$,

$$gr_k(K_3 : S_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Gallai-Ramsey number for path with an extra edge

- The precise Gallai-Ramsey number for P_t^+ are also obtained.

Theorem 2.24 (Wang, Mao, Magnant, Zou)

For $t \geq 4$ and $k \geq 1$,

$$gr_k(K_3 : P_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \textit{if } k \textit{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \textit{if } k \textit{ is odd.} \end{cases}$$

Outline

1. Introduction
   - Classical Ramsey number
   - Gallai Ramsey number under K_3-coloring
   - Gallai Ramsey number under S_3^+-coloring
2. Main results
   - Books
   - Wheels
   - Fans
   - Stars with extra independent edges
   - Odd cycles
   - Two Classes of Unicyclic Graphs
   - Results for S_3^+-coloring

Stars

- S. Fujita, C. Magnant, Extensions of Gallai-Ramsey results, J. Graph Theory 70(4) (2012), 404–426 proposed the following conjecture for star graphs.

Conjecture 2.2

For $k \geq 4$,

$$gr_k(S_3^+; K_{1,t}) = 3t - 2k + 4.$$

A tree of order five

Theorem 2.25

For $k \geq 1$,

$$gr_k(K_3; T_1) = k + 4,$$

where T_1 is a tree obtained from a path P_4 by adding a pendant edge on one of the internal vertices in P_4.

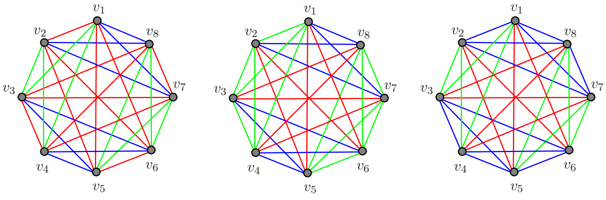
Theorem 2.26 (Mao, Su, Wang, Magnant)

For $k \geq 1$,

$$gr_k(S_3^+; T_1) = \begin{cases} k+4, & \text{if } k \neq 3; \\ 9, & \text{if } k = 3, \end{cases}$$

Sketch of the Proof

- To show $gr_3(S_3^+; T_1) \geq 9$, we have the following examples.



(a) (b) (c)

The End

Thank you for your attention !